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Asymptotics for Orthogonal Polynomials On and Off the Essential Spectrum

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We assume that the coefficients in the three term recurrence relation for orthogonal polynomials form sequences of bounded variation and we show that the asymptotic behavior of the orthogonal polynomials on the essential spectrum is closely related to the asymptotic behavior in the complex plane (off the essential spectrum), proving a conjecture of Máté, Nevai and Totik [*Constr. Approx.* **1** (1985), 231–248]. © 1988 Academic Press, Inc.

I. INTRODUCTION

Let $\{p_n(x): n=0, 1, 2,...\}$ be a sequence of polynomials satisfying the three term recurrence relation

$$xp_n(x)=a_{n+1}\,p_{n+1}(x)+b_n\,p_n(x)+a_n\,p_{n-1}(x) \tag{1.1}$$

with initial conditions $p_{-1}(x)=0$ and $p_0(x)=1$. We assume that the recurrence coefficients satisfy

$$\lim_{n\to\infty} a_n=1/2, \qquad \lim_{n\to\infty} b_n=0, \tag{1.2}$$

$$\sum_{n=1}^{\infty} \{|a_{n+1}-a_n|+|b_{n+1}-b_n|\}<\infty. \tag{1.3}$$

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It is well known that there exists a measure α such that

$$\int_{-\infty}^{+\infty} p_n(x)\, p_m(x)\, d\alpha(x) = \delta_{m,n},\ m, n \geqslant 0,$$

and, by a result of Blumenthal [1], Condition (1.2) implies that $[-1, 1] \subset \operatorname{supp}(\alpha)$ and $\operatorname{supp}(\alpha) \backslash [-1, 1]$ is a bounded denumerable set whose only possible points of accumulation are 1 and -1. The infinite Jacobi matrix defined by the recurrence coefficients a_n and b_n is a linear operator in ℓ_2. The interval $[-1, 1]$ is referred to as the essential spectrum while the points in $\operatorname{supp}(\alpha) \backslash [-1, 1]$ are the eigenvalues of the Jacobi matrix (Kato [2, pp. 243, 518]). Máté and Nevai [3] have shown that (1.3) implies that α is absolutely continuous in $(-1, 1)$ and that $\alpha'(x) > 0$ for $-1 < x < 1$. Máté, Nevai and Totik [4] have studied the asymptotic behavior of the orthogonal polynomials. In order to state their results we need to introduce some notation. Let

$$\rho(z) = z + \sqrt{z^2 - 1}, \tag{1.4}$$

where the square root is such that $|\rho(z)| > 1$ when $z \in \mathbb{C} \backslash [-1, 1]$. Then ρ is analytic in $\mathbb{C} \backslash [-1, 1]$. For $x \in (-1, 1)$ we define $\rho(x)$ by

$$\rho(x) = \lim_{y \to 0+} \rho(x + iy)$$

so that $|\rho(x)| = 1$ and $0 < \arg \rho(x) < \pi$ for $-1 < x < 1$. Next we let $t_{1,n}$ and $t_{2,n}$ be the zeros of $a_{n+1} t^2 + (b_n - z)\, t + a_n$, more precisely

$$t_{1,n} = t_{1,n}(z) = \sqrt{\frac{a_n}{a_{n+1}}}\, \rho\left(\frac{x - b_n}{2\sqrt{a_n a_{n+1}}}\right)$$
$$t_{2,n} = t_{2,n}(z) = \sqrt{\frac{a_n}{a_{n+1}}}\, \rho\left(\frac{x - b_n}{2\sqrt{a_n a_{n+1}}}\right)^{-1}.$$

The main results of Máté, Nevai and Totik can then be summarized in:

THEOREM 1. (a) *Let K_1 be a compact set in $\mathbb{C} \backslash [-1, 1]$, then there exists a function g_1 in $\mathbb{C} \backslash [-1, 1]$ such that $g_1(z) \prod_{k=1}^{N} t_{1,k}$ is analytic in K_1 for N large enough, $g_1(z) = 0$ if and only if $z \in \operatorname{supp}(\alpha) \backslash [-1, 1]$, and*

$$\lim_{n \to \infty} \frac{p_n(z)}{\prod_{k=1}^{n} t_{1,k}} = \frac{g_1(z)}{2\sqrt{z^2 - 1}}$$

holds uniformly for $z \in K_1$.

(b) *Let K_2 be a compact set in $(-1, 1)$, then there exists a non-vanishing continuous function g_2 in $(-1, 1)$ such that*

$$\lim_{n\to\infty} \left\{\sqrt{1-x^2}\, p_n(x) - |g_2(x)| \prod_{k=1}^{n} |t_{1,k}| \times \sin\left(\sum_{k=1}^{n} \arg t_{1,k} + \arg g_2(x)\right)\right\} = 0$$

holds uniformly for $x \in K_2$.

Máté, Nevai and Totik [4] have conjectured that the functions g_1 and g_2 are closely related, namely

$$\lim_{y\to 0+} g_1(x+iy) = g_2(x), \qquad x \in (-1, 1). \tag{1.5}$$

We will present two proofs of this conjecture.

II. First Proof

We introduce the functions

$$\phi_n(z) = p_n(z) - t_{2,n}\, p_{n-1}(z), \qquad n = 0, 1, 2, ... \tag{2.1}$$

It is shown in [3, 4] that they satisfy

$$\phi_{n+1}(z) - t_{1,n}\phi_n(z) = (t_{2,n} - t_{2,n+1})\, p_n(z). \tag{2.2}$$

From this one can investigate the ratio ϕ_{n+1}/ϕ_n.

LEMMA 1. *Let* $\varepsilon_n = |a_{n+1} - a_n| + |b_{n+1} - b_n|$ *and* $K_\varepsilon = \{z \in \mathbb{C}: 0 \leqslant \operatorname{Im} z \leqslant 1, |\operatorname{Re} z| \leqslant \varepsilon\}$ *where* $0 < \varepsilon < 1$, *then*

$$\frac{\phi_{n+1}(z)}{\phi_n(z)} - t_{1,n}(z) = O(\varepsilon_n + \varepsilon_{n+1}) \tag{2.3}$$

holds uniformly on K_ε and consequently

$$\lim_{n\to\infty} \frac{\phi_{n+1}(z)}{\phi_n(z)} = \rho(z) \tag{2.4}$$

uniformly on K_ε.

Proof. From (2.2) we derive

$$\frac{\phi_{n+1}(z)}{\phi_n(z)} - t_{1,n}(z) = (t_{2,n} - t_{2,n+1}) \frac{p_n(z)}{\phi_n(z)} \tag{2.5}$$

and from Eq. (14) in [4] we borrow

$$|t_{i,n} - t_{i,n+1}| = O(\varepsilon_n + \varepsilon_{n+1}) \qquad (i = 1, 2) \tag{2.6}$$

uniformly in K_ε. Clearly

$$\frac{p_n(z)}{\phi_n(z)} = \frac{a_{n+1}}{a_n} t_{1,n} \left\{ \frac{a_{n+1}}{a_n} t_{1,n} - \frac{p_{n-1}(z)}{p_n(z)} \right\}^{-1}, \tag{2.7}$$

where we have used (2.1) and $t_{2,n}^{-1} = (a_{n+1}/a_n)\, t_{1,n}$. We can decompose the ratio $p_{n-1}(z)/p_n(z)$ into partial fractions,

$$\frac{p_{n-1}(z)}{p_n(z)} = \sum_{j=1}^{n} \frac{a_{j,n}}{z - x_{j,n}},$$

where $\{x_{j,n}\}$ are the zeros of p_n and $\{a_{j,n}\}$ are positive real numbers (Szegö [6, p. 47]). Let $z = x + iy$ where $|x| \leqslant \varepsilon$ and $0 \leqslant y \leqslant 1$, then

$$\operatorname{Im} \frac{p_{n-1}(x+iy)}{p_n(x+iy)} = -\sum_{j=1}^{n} \frac{y}{(x - x_{j,n})^2 + y^2} a_{j,n} \leqslant 0$$

(actually this is a general result that holds for every Stieltjes transform of a positive measure) hence

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} t_{1,n} - \frac{p_{n-1}(z)}{p_n(z)} \right| &\geqslant \left| \operatorname{Im} \left\{ \frac{a_{n+1}}{a_n} t_{1,n} - \frac{p_{n-1}(x+iy)}{p_n(x+iy)} \right\} \right| \\ &\geqslant \frac{a_{n+1}}{a_n} \operatorname{Im} t_{1,n} \end{aligned}$$

and if we use this inequality in (2.7) we obtain

$$\left| \frac{p_n(z)}{\phi_n(z)} \right| \leqslant \frac{|t_{1,n}|}{\operatorname{Im} t_{1,n}} = O(1)$$

uniformly in K_ε. Together with (1.3) and (2.6) this proves the desired result. ∎

It is well known (Nevai [5]) that (1.2) implies

$$\lim_{n\to\infty} \frac{p_{n+1}(z)}{p_n(z)} = \rho(z)$$

holds uniformly on every compact set in $\mathbb{C}\backslash\text{supp}(\alpha)$. This ratio of orthogonal polynomials does not converge on $[-1, 1]$ since the zeros of the orthogonal polynomials are dense in $[-1, 1]$ so that p_n has oscillatory behavior on the essential spectrum. The main advantage of using the functions ϕ_n is that the ratio ϕ_{n+1}/ϕ_n can be handled on $(-1, 1)$.

Introduce the functions

$$\psi_n(z) = \phi_n(z) - t_{2,n-1}\phi_{n-1}(z), \qquad n = 1, 2, ... \tag{2.8}$$

Their asymptotic behavior is given by:

LEMMA 2. *There exists a continuous function G on K_ε such that*

$$\lim_{n\to\infty} \frac{\psi_{n+1}(z)}{\prod_{k=1}^n t_{1,k}} = G(z)$$

holds uniformly on K_ε.

Proof. From (2.8) one immediately finds

$$\frac{\psi_{n+1}(z)}{t_{1,n}\psi_n(z)} = \frac{\phi_n(z)}{t_{1,n}\phi_{n-1}(z)} \frac{(\phi_{n+1}(z)/\phi_n(z)) - t_{2,n}}{(\phi_n(z)/\phi_{n-1}(z)) - t_{2,n-1}}. \tag{2.9}$$

Using Lemma 1 gives

$$\frac{\phi_{n+1}(z)}{\phi_n(z)} - t_{2,n} = O(\varepsilon_n + \varepsilon_{n+1}) + (t_{1,n} - t_{2,n}) \tag{2.10}$$

and

$$\frac{\phi_n(z)}{t_{1,n}\phi_{n-1}(z)} = 1 + O(\varepsilon_{n-1} + \varepsilon_n) \tag{2.11}$$

and in combining (2.9)–(2.11) together with (2.6) we obtain

$$\frac{\psi_{n+1}(z)}{t_{1,n}\psi_n(z)} = 1 + O(\varepsilon_{n+1} + \varepsilon_n + \varepsilon_{n-1}).$$

Use (1.3) to conclude that

$$\frac{\psi_{n+1}(z)}{\prod_{k=1}^n t_{1,k}\psi_1(z)} = \prod_{k=1}^n \frac{\psi_{k+1}(z)}{t_{1,k}\psi_k(z)}$$

converges uniformly on K_ε as n tends to infinity. This proves the lemma since every function on the left hand side is continuous in K_ε. ∎

From Lemma 1 it follows that

$$\lim_{n\to\infty} \frac{\psi_{n+1}(z)}{\phi_n(z)} = \rho(z) - \frac{1}{\rho(z)} = 2\sqrt{z^2-1}$$

uniformly on K_ε, so that Lemma 2 implies

$$\lim_{n\to\infty} \frac{\phi_n(z)}{\prod_{k=1}^{n-1} t_{1,k}} = g(z),$$

where $g(z) = \rho(z)\, G(z)/(2\sqrt{z^2-1})$ is continuous in K_ε. If we take a compact set K_1 in $K_\varepsilon \backslash [-\varepsilon, \varepsilon]$ then on K_1

$$\lim_{n\to\infty} \frac{p_n(z)}{\phi_n(z)} = \frac{\rho(z)}{2\sqrt{z^2-1}}$$

so that on K_1

$$\lim_{n\to\infty} \frac{p_n(z)}{\prod_{k=1}^{n} t_{1,k}} = \frac{g(z)}{2\sqrt{z^2-1}}.$$

Compare this with (a) in Theorem 1, then it follows that the function g_1 in Theorem 1 coincides with g in $K_\varepsilon \backslash [-\varepsilon, \varepsilon]$. On the other hand we have on $[-\varepsilon, \varepsilon]$

$$\operatorname{Im} \phi_n(x) = \operatorname{Im} \left\{ g(x) \prod_{k=1}^{n-1} t_{1,k} \right\} + o(1)$$

and since $p_n(x)$ and $p_{n-1}(x)$ are real and $|\rho(x)| = 1$ on $[-\varepsilon, \varepsilon]$ this gives

$$-(\operatorname{Im} t_{2,n})\, p_{n-1}(x) = |g(x)| \prod_{k=1}^{n-1} |t_{1,k}| \times \sin\left(\sum_{k=1}^{n-1} \arg t_{1,k} + \arg g(x) \right) + o(1)$$

from which one easily obtains

$$\lim_{n\to\infty} \left\{ \sqrt{1-x^2}\, p_n(x) - |g(x)| \prod_{k=1}^{n} |t_{1,k}| \times \sin\left(\sum_{k=1}^{n} \arg t_{1,k} + \arg g(x) \right) \right\} = 0.$$

Comparison with statement (b) of Theorem 1 shows that the function g_2 in Theorem 1 coincides with g on $[-\varepsilon, \varepsilon]$. Since g is continuous in K_ε it then follows that (1.5) is true, which provides the conjecture of Máté, Nevai and Totik.

III. Second Proof

We will use some techniques from scattering theory. The three term recurrence relation (1.1) can be written as

$$p_{k+1}(x) + \frac{a_k}{a_{k+1}} p_{k-1}(x) - (t_{1,k} + t_{2,k})\, p_k(x) = 0. \tag{3.1}$$

Introduce the discrete Green's function $G(k, m)$ as the solution of

$$\frac{a_{k+1}}{a_{k+2}} G(k+1, m) + G(k-1, m) - (t_{1,k+1} + t_{2,k})\, G(k, m) = \delta_{k,m} \tag{3.2}$$

with $G(k, m) = 0$ $(k \geqslant m)$. One can check that

$$G(k, m) = \sum_{i=k+1}^{m} \left(\prod_{j=i+1}^{m} t_{1,j} \right) \left(\prod_{j=k+1}^{i-1} t_{2,j} \right) \tag{3.3}$$

since this array satisfies

$$\frac{a_{k+1}}{a_{k+2}} G(k+1, m) - t_{1,k+1}\, G(k, m) = -\prod_{j=k+1}^{m} t_{1,j} \tag{3.4}$$

from which (3.2) is easily derived (an empty product is always taken as one). This Green's function is useful in studying the orthogonal polynomials, since if one multiplies (3.1) by $G(k, m)$ and (3.2) by $p_k(x)$ and subtracts the obtained relations one gets

$$\begin{aligned} p_k(x)\, \delta_{k,m} = {} & (t_{1,k} - t_{1,k+1})\, p_k(x)\, G(k, m) + p_k(x)\, G(k-1, m) \\ & - p_{k+1}(x)\, G(k, m) + \frac{a_{k+1}}{a_{k+2}} p_k(x)\, G(k+1, m) \\ & - \frac{a_k}{a_{k+1}} p_{k-1}(x)\, G(k, m). \end{aligned}$$

Summing from $k = 0$ to $k = m$ gives after some straightforward manipulations

$$p_m(x) = G(-1, m) + \sum_{k=0}^{m-1} (t_{1,k} - t_{1,k+1})\, p_k(x)\, G(k, m) \tag{3.5}$$

which is a discrete integral equation for $\{p_k(x)\}$. Let us first obtain a bound for the Green's function.

LEMMA 3. *Define* $\hat{G}(k, m) = (\prod_{i=k}^{m} t_{1,i}^{-1})\, G(k, m)$, *and assume that* (1.2) *holds, then for every compact set* K *in* $\mathbb{C}\setminus\{-1, 1\}$ *there exist constants* C *and* D (*depending on* K) *such that*

$$|\hat{G}(k, m)| \leqslant C \exp\left\{D \sum_{i=k}^{m} \varepsilon_i\right\}. \tag{3.6}$$

Proof. From (3.4) we obtain

$$\frac{a_{k+1}}{a_{k+2}} \{\hat{G}(k+1, m) - t_{2,k+1}^{-1}\, t_{1,k}\, \hat{G}(k, m)\} = -1 \tag{3.7}$$

so that

$$\frac{a_{k+1}}{a_{k+2}} \left\{\hat{G}(k+1, m) - \frac{t_{1,k}}{t_{2,k+1}} \hat{G}(k, m)\right\}$$
$$= \frac{a_{k+2}}{a_{k+3}} \left\{\hat{G}(k+2, m) - \frac{t_{1,k+1}}{t_{2,k+2}} \hat{G}(k+1, m)\right\}.$$

Thus one finds

$$\left\{\frac{a_k}{a_{k+1}} \hat{G}(k, m) - \frac{a_{k+1}}{a_{k+2}} \hat{G}(k+1, m)\right\}$$
$$= t_{2,k} \left\{t_{1,k+2} - \frac{a_{k+1}}{a_{k+2}} t_{2,k}^{-1}\right\} \hat{G}(k+1, m)$$
$$+ \frac{t_{2,k}}{t_{1,k+1}} \left\{\frac{a_{k+1}}{a_{k+2}} \hat{G}(k+1, m) - \frac{a_{k+2}}{a_{k+3}} \hat{G}(k+2, m)\right\}.$$

Iteration of this last equation gives

$$\left\{\frac{a_k}{a_{k+1}} \hat{G}(k, m) - \frac{a_{k+1}}{a_{k+2}} \hat{G}(k+1, m)\right\} = R(k, m) \frac{a_{m-1}}{a_m} \frac{1}{t_{1,m-1}\, t_{1,m}}$$
$$+ \sum_{i=k+1}^{m-1} t_{2,i-1} \left\{t_{1,i+1} - \frac{a_i}{a_{i+1}} t_{2,i-1}^{-1}\right\} R(k, i)\, \hat{G}(i, m), \tag{3.8}$$

where $R(k, m) = \prod_{j=k+1}^{m-1} t_{2,j-1}/t_{1,j}$. From (3.7) one also obtains

$$\left\{\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} \frac{t_{1,k}}{t_{2,k+1}}\right\} \hat{G}(k, m) = -1 + \left\{\frac{a_k}{a_{k+1}} \hat{G}(k, m) - \frac{a_{k+1}}{a_{k+2}} \hat{G}(k+1, m)\right\}$$

and inserting (3.8) gives

$$\left\{\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} \frac{t_{1,k}}{t_{2,k+1}}\right\} \hat{G}(k, m) = -1 + \frac{a_{m-1}}{a_m} \frac{1}{t_{1,m-1} t_{1,m}} R(k, m)$$
$$+ \sum_{i=k+1}^{m-1} t_{2,i-1} \left\{t_{1,i+1} - \frac{a_i}{a_{i+1}} t_{2,i-1}^{-1}\right\} R(k, i)\, \hat{G}(i, m).$$

Write

$$\left\{\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} \frac{t_{1,k}}{t_{2,k+1}}\right\} \hat{G}(k, m) = \sum_{j=0}^{\infty} g_j(k, m),$$

where

$$g_0(k, m) = -1 + \frac{a_{m-1}}{a_m} \frac{1}{t_{1,m-1}\, t_{1,m}} R(k, m)$$
$$g_j(k, m) = \sum_{i=k+1}^{m-1} t_{2,i-1} \frac{t_{1,i+1} - (a_i/a_{i+1})\, t_{2,i-1}^{-1}}{(a_i/a_{i+1}) - (a_{i+1}/a_{i+2})(t_{1,i}/t_{2,i+1})} \times R(k, i)\, g_{j-1}(i, m).$$

Since $\{a_n\}$ is a bounded sequence and $|t_{2,j}/t_{1,j}| \leqslant 1$ there exists a constant C_1 such that $|g_0(k, m)| \leqslant C_1$. By standard iterative manipulations one therefore obtains

$$\left|\left\{\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} \frac{t_{1,k}}{t_{2,k+1}}\right\} \hat{G}(k, m)\right|$$
$$\leqslant C_1 \exp\left\{\sqrt{\frac{a_k}{a_{k+1}}} \sum_{i=k+1}^{m-1} \frac{|t_{1,i+1} - (a_i/a_{i+1})\, t_{2,i-1}^{-1}|}{|(a_i/a_{i+1}) - (a_{i+1}/a_{i+2})(t_{1,i}/t_{2,i+1})|}\right\}. \tag{3.9}$$

Let K be a compact set in $\mathbb{C} \backslash \{-1, 1\}$. Since

$$\lim_{k \to \infty} \left\{\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} \frac{t_{1,k}}{t_{2,k+1}}\right\} = 1 - \rho^2$$

and K is a positive distance away from 1 and -1, there exists a constant C_2 and an integer k_0 such that

$$\frac{1}{|(a_k/a_{k+1}) - (a_{k+1}/a_{k+2})(t_{1,k}/t_{2,k+1})|} \leqslant C_2$$

holds on K for $k \geqslant k_0$ (the denominator of the left hand side may be zero for a finite number of k). The bound (3.6) then follows for $k \geqslant k_0$ by observing that

$$\left| t_{1,i+1} - \frac{a_i}{a_{i+1}} t_{2,i-1)}^{-1} \right| = O(\varepsilon_{i+1} + \varepsilon_i + \varepsilon_{i-1})$$

which can be deduced from (2.6). By the recurrence relation (3.2) one easily finds that (3.6) is also valid for $k_0 - 1$ and by induction for every k such that $-1 \leqslant k < k_0$. ∎

The bound on the Green's function can be used to obtain a bound on the orthogonal polynomials.

LEMMA 4. *Define* $\hat{p}_m(x) = (\prod_{i=1}^m t_{1,i}^{-1})\, p_m(x)$, *and assume that* (1.2) *and* (1.3) *hold, then for every compact set* K *in* $\mathbb{C} \setminus \{-1, 1\}$ *there exist constants* A *and* B *such that*

$$|\hat{p}_m(x)| \leqslant A \exp\left\{ B \sum_{k=1}^{m} \varepsilon_k \right\}. \tag{3.10}$$

Proof. From (3.5) we obtain

$$\hat{p}_m(x) = \hat{G}(-1, m) + \sum_{k=0}^{m-1} t_{1,k}(t_{1,k} - t_{1,k+1})\, \hat{p}_k(x)\, \hat{G}(k, m).$$

Write

$$\hat{p}_m(x) = \sum_{i=0}^{\infty} w_i(m),$$

where

$$w_0(m) = \hat{G}(-1, m)$$

$$w_i(m) = \sum_{k=0}^{m-1} t_{1,k}(t_{1,k} - t_{1,k+1})\, \hat{G}(k, m)\, w_{i-1}(k),$$

then from Lemma 3 we find a constant A such that

$$|w_0(m)| \leqslant A.$$

By induction (iteration) we find

$$|\hat{p}_m(x)| \leqslant A \exp\left\{ A \sum_{k=0}^{m-1} |t_{1,k}|\, |t_{1,k} - t_{1,k+1}| \right\}$$

and the result follows from (2.6). ∎

Next we will show that the functions $\{\phi_n(z)\}$ converge as n tends to infinity.

LEMMA 5. *There exists a function g in $\mathbb{C}\backslash\{-1, 1\}$ such that*

$$\lim_{n\to\infty} \frac{\phi_n(z)}{\prod_{k=1}^{n-1} t_{1,k}} = g(z)$$

uniformly in every compact set $K \subset \mathbb{C}\backslash\{-1, 1\}$.

Proof. Let $\hat{\phi}_n(z) = (\prod_{i=0}^{n-1} t_{1,i}^{-1})\, \phi_n(z)$ then from (2.2) we find

$$\hat{\phi}_{n+1}(z) - \hat{\phi}_n(z) = (t_{2,n} - t_{2,n+1})\, \hat{p}_n(z)$$

which gives

$$\hat{\phi}_n(z) - \hat{\phi}_m(z) = \sum_{i=m}^{n-1} (t_{2,i} - t_{2,i+1})\, \hat{p}_i(z).$$

Use the bound (3.10) to find

$$\begin{aligned}|\hat{\phi}_n(z) - \hat{\phi}_m(z)| &\leqslant A \exp\left\{B \sum_{k=1}^{\infty} \varepsilon_k\right\} \sum_{i=m}^{n-1} |t_{2,i} - t_{2,i+1}| \\ &= O\left(\sum_{i=m}^{n} \varepsilon_i\right).\end{aligned}$$

By (1.3) it follows that $\{\phi_n\}$ is a Cauchy sequence in every compact set $K \subset \mathbb{C}\backslash\{-1, 1\}$, from which the result follows. ∎

Every function $\hat{\phi}_n(z)$ is continuous in a compact set of the upper half plane $\{\operatorname{Im} z \geqslant 0\}$. This means that the function g is continuous in every compact set $K \subset \{\operatorname{Im} z \geqslant 0\}\backslash\{-1, 1\}$ and we can repeat the reasoning in Section II to prove the conjecture (1.5) of Máté, Nevai, and Totik.

IV. Concluding Remarks

The asymptotic behavior of the orthogonal polynomials under Conditions (1.2) and (1.3) gives much information about the measure α and the weight function $\alpha'(x)$ on $(-1, 1)$. Máté and Nevai [3] have shown that for $x \in (-1, 1)$

$$0 < |g(x)| \prod_{k=1}^{\infty} |t_{1,k}| = \sqrt{\frac{2}{\pi} \frac{\sqrt{1-x^2}}{\alpha'(x)}} < \infty.$$

Their analysis involved the investigation of the orthogonal polynomials on $(-1, 1)$ which is a difficult task since the zeros of the orthogonal polynomials are dense in $[-1, 1]$. It is easier to investigate the polynomials in $\mathbb{C}\backslash[-1, 1]$ and by the result above this still gives all the information needed to find $\alpha'(x)$. One only fixed the real part of z at $x \in (-1, 1)$ and then lets the (positive) imaginary part of z tend to zero to find $g(x)$, giving the weight function.

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